How to Beat the Casinos

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Abstract

This report aims to explore certain betting strategies and how they can be utilised in a casino environment to the gamblers advantage. The main problem with gambling is to find an opportunity with a positive expectation so your wealth, in the long run, can grow. I will explore some of the methods to achieve this and how to maximise profit once this positive expectation is achieved. It will focus on the Kelly Criterion, a strategy which tells the gambler a specific percentage of their total capital to bet in order to maximise the expected growth.

Contents

‘This piece of work is a result of my own work except where it forms an assessment based on group project work. In the case of a group project, the work has been prepared in collaboration with other members of the group. Material from the work of others not involved in the project has been acknowledged and quotations and paraphrases suitably indicated.’

Chapter 1

Introduction

Gambling is one of the oldest activities in the world with its history dating back thousands of years. It has transformed from very simple dice-like games in ancient mesopotamia to the billion dollar business that prop up many economies throughout the world.

Gambling is thought to have started in ancient mesopotamia using the knuckle or ankle bones as primitive dice as far back as 6000BC. By 3000BC dice-like games were being played across the world by early Chinese, European and Middle Eastern civilisations. The Egyptians were the first people to invent the cubed dice that we recognise today. Around 1000BC the Chinese were gambling with tiles and gambling had become a day to day pastime, woven into everyday life. Throughout the next thousand years it became popular to gamble on outcomes of different events or competitions; betting on fights or races between animals or humans alike. This is shown throughout the early Olympic games as gambling was rife. Throughout the Roman Empire gambling was also common and Emperor Claudius redesigned his carriage to allow more room to play dice . It was also stated that all children had to learn how to gamble and throw dice. [2][1]

The next big breakthrough, around 100AD, was the invention of paper. This allowed a great increase in the variety of different games available to bet on, through the innovation of playing cards. Over the next two thousand years the popularity of gambling grew until large casino cities were built in 20th Century after gambling was legalised in many US states. With many foreign travellers coming to these new casinos bringing different varieties of the games they played, and with the casinos trying to tweak the rules to put the odds in their favour there soon became the many different types of casino games that are available today. [2] [3]

As gambling evolved so did the strategies associated with them. Evolving from very simple strategies from betting on instinct to betting the amount that mathe-

[1] Numbers in square brackets refer to the bibliography

matically increases your chances of winning. I will start by introducing simple stopping strategies and then move onto having a detailed look at the Kelly Criterion. In chapter 4 I will look at different betting strategies, then comparing them with the Kelly Criterion. I will finally look at physical strategies that actually change the probability of winning and how it can be used in the real world to make money.

1.1 Definitions

The following are definitions of terms used throughout this report specific to the gambling world;

- **Gambling** A gamble is the risk on a certain outcome occurring, with a favourable result.
- **Bet/Wager** The action of actually placing money on a specific gamble.
- **Stake** The amount of money placed on a certain event occurring.
- **Betting Strategy** Information that tells the player how much to bet in certain situations.
- **Odds** The amount of money paid if the bet is successful in relation to the stake. i.e. 5-1 pays 5 units profit for each 1 unit bet.
- **House** The casino offering the bet will be called the house.
- **Capital** The total amount of money the player is willing to bet with.
- **Favourable Game** A favourable game is game with positive expectation e.g. a game paying even (1-1) odds but the probability of winning $p > 0.5$.
- **Gamblers Ruin** A player has reached gamblers ruin when their capital has reached zero. They are bankrupt.

1.2 Assumptions

The following are some of the assumptions used throughout this report;

- The aim of every bet is to be successful. This depends on the situation whether it is to minimise the chance of ruin or maximise the capital.
- All games using dice or coins will be fair unless otherwise stated. When using a coin a success will be a head and a failure will be a tail. In cards games a standard 52 card deck will be used.

- The probability of success is $p < 1$ as if $p = 1$ then no gamble would exist and the chance of ruin would be 0. The probability of failure is $q = 1 - p$.

- Once the player reaches gamblers ruin they stop. They cannot get more money from a loan.

- The aim of the player is to find favourable games. This may provide an opportunity for arbitrage.

Chapter 2

Simple Probability Games

In this chapter I will be looking at some very simple probability games and how some of the outcomes of these games are counter intuitive to common sense. I will also be considering some of the simplest strategies in gambling, a stopping strategy and a strategy to win a specific game; Penney's Game. For all examples in this chapter I will be using a fair 6-sided die with the probability of any side coming up as 1/6 and a fair coin with heads or tails equally likely to arise.

2.1 Stopping Strategies

A stopping strategy can be considered a strategy to a game where you stop playing that game after a certain event. For example a specified amount of time or when you have a certain amount of money. Stopping strategies can be a very simple way to try to minimise loses or a good way to leave a game with profit. The best way to explain a stopping strategy is by an example [5].

2.1.1 Example

Imagine a very simple probability game. A player rolls a die and wins £2 if they roll a 1 or 2, but loses £1 if they roll anything else i.e. 3,4,5 or 6. Therefore about a third of the time the player will win £2 and two thirds of the time will lose £1. The game is intuitively fair as on average the player wins double what they lose, half the number of times. Therefore in the long run you would expect your total capital to remain the same.

The game is played repeatedly with exactly the same rules and each roll of the die independent of previous rolls. A stopping strategy is set up declaring that the player will only stop playing the game when they reach a certain amount money or reach gamblers ruin and have no capital left. For this example the player will start with £5. The stopping strategy will be that the game stops when the player has doubled their money i.e. their capital is 0, 10 or 11. A recurence relation can be set up to compare probabilities.

$$P_N^{(k)} = \frac{1}{3}P_{N+2}^{(k)} + \frac{2}{3}P_{N-1}^{(k)}$$

Where;
N=The amount of money you start with i.e. N=5
k=The amount you end with i.e. 0,10,11
$P_N^{(k)}$ = The probability of ending up with k, when starting with N

This relationship comes from the fact that the probability of going from N to N+2 is 1/3 and the probability of going from N to N-1 is 2/3.

For this example N=5

$$P_5^{(k)} = \frac{1}{3}P_7^{(k)} + \frac{2}{3}P_4^{(k)}$$

We look for a solution in the form $P_N^{(k)} = x^N$ with $x \in \mathbb{R}$

$$\begin{aligned} x^5 &= \frac{1}{3}x^7 + \frac{2}{3}x^4 \\ x &= \frac{1}{3}x^3 + \frac{2}{3} \end{aligned}$$

This is the characteristic equation for the recurence relation.
You can write this as a continuous function of x;

$$\begin{aligned} f(x) &= \frac{1}{3}x^3 - x + \frac{2}{3} \\ f(x) &= \frac{1}{3}(x-1)^2(x+2) \end{aligned}$$

It can be shown that when the characteristic polynomial has a double root at x=r, then $P_N^{(k)} = N.r^N$ is another solution. In this example the general solutions are linear combinations of $1^N = 1$, N.$1^N = N$ and $(-2)^N$. Different combinations of these solutions will provide the correct answer for each k.

$$\begin{aligned} P_{N_1}^{(0)} &= x_{11} + x_{12}.N_1 + x_{13}(-2)^{N_1} \\ P_{N_2}^{(10)} &= x_{21} + x_{22}.N_2 + x_{23}(-2)^{N_2} \\ P_{N_3}^{(11)} &= x_{31} + x_{32}.N_3 + x_{33}(-2)^{N_3} \end{aligned}$$

This can be written in matrix notation;

$$\begin{pmatrix} P_{N_1}^{(0)} \\ P_{N_2}^{(10)} \\ P_{N_3}^{(11)} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ N_1 & N_2 & N_3 \\ (-2)^{N_1} & (-2)^{N_2} & (-2)^{N_3} \end{pmatrix}$$

To work out the numbers x_{ij} you need to put the numbers N_1, N_2, N_3 so that you already know the probabilities $P_{N_1}^{(0)}, P_{N_2}^{(10)}, P_{N_3}^{(11)}$. If you are stopping at 0, the only number to start at that ensures you certainly stop at 0 is to start at 0. This is the same for stopping at 10 or 11. Hence $P_0^{(0)} = 1, P_{10}^{(10)} = 1, P_{11}^{(11)} = 1$ and $N_1 = 0, N_2 = 10, N_3 = 11$

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 10 & 11 \\ 1 & 1024 & -2048 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now find the inverse of the middle matrix;

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 10 & 11 \\ 1 & 1024 & -2048 \end{pmatrix}^{-1} = \frac{1}{31743} \begin{pmatrix} 31744 & -3072 & -1 \\ -11 & 2049 & 11 \\ 10 & 1023 & -10 \end{pmatrix}$$

Now we have the values for x_{ij} so can find the values $P_N^{(k)}$

$$\begin{aligned} P_N^{(0)} &= (31744 - 3072N - (-2)^N)/31743 \\ P_N^{(10)} &= (-11 + 2049N + 11(-2)^N)/31743 \\ P_N^{(11)} &= (10 + 1023N - 10(-2)^N)/31743 \end{aligned}$$

So for the example where N=5;

$$\begin{aligned} P_5^{(0)} &= (31744 - 15360 + 32)/31743 = 0.51715 \\ P_5^{(10)} &= (-11 + 10245 - 352)/31743 = 0.31131 \\ P_5^{(11)} &= (10 + 5115 + 320)/31743 = 0.17153 \end{aligned}$$

From this you can see that although intuitively you would assume that it is a fair game and the probability of winning or losing would be equal, it is clearly not. The probability of losing all your money is slightly higher than winning at least double. I.e gambler's ruin about 52% of the time, but win at least double about 48% of the time. You can put any number $0 < N < 9$ with the same strategy of stopping at 0,10 or 11 and work out the probabilities of ruin or success. I calculated the other probabilities for the other starting amounts and are in the following table:

Table 2.1:

N	k=0	k=10	k=11
1	0.9033	0.0635	0.0332
2	0.8063	0.1301	0.0635
3	0.7099	0.1905	0.0995
4	0.6124	0.2634	0.1242
5	0.5172	0.3113	0.1715
6	0.4173	0.4091	0.1735
7	0.3266	0.4071	0.2663
8	0.2177	0.6047	0.1776

Now I will construct my own more complicated, generalised example. In my example, you can win £1 with probability 1/3, £2 with probability 1/9, but lose £1 with probability 5/9. As before it is still looks to be a fair game, with expected value of each roll equal to 0. The player will start with £N and stop at minimum of £M.

$$P_N^{(M)} = \frac{1}{3}P_{N+1}^{(k)} + \frac{1}{9}P_{N+2}^{(k)} + \frac{5}{9}P_{N-1}^{(k)}$$

As before take a guess that $P_N^{(k)} = x^N$ with $x \in R$

$$\begin{aligned} x^N &= \frac{1}{3}x^{N+1} + \frac{1}{9}x^{N+2} + \frac{5}{9}x^{N-1} \\ x &= \frac{1}{3}x^2 + \frac{1}{9}x^3 + \frac{5}{9} \end{aligned}$$

This is the characteristic equation for the recurence relation.
You can write this as a continuous function of x;

$$\begin{aligned} f(x) &= \frac{1}{9}x^3 + \frac{1}{3}x^2 - x + \frac{5}{9} \\ f(x) &= \frac{1}{9}(x-1)^2(x+5) \end{aligned}$$

General solutions can be constructed from previous analysis and are: 1, N, $(-5)^N$.

In this case you stop, when you are ruined i.e at 0 or when you reach £M. Therefore the 3 possible outcomes are 0, M and M+1.

$$\begin{aligned} P^{(0)}_{N_1} &= x_{11} + x_{12}.N_1 + x_{13}(-5)^{N_1} \\ P^{(M)}_{N_2} &= x_{21} + x_{22}.N_2 + x_{23}(-5)^{N_2} \\ P^{(M+1)}_{N_3} &= x_{31} + x_{32}.N_3 + x_{33}(-5)^{N_3} \end{aligned}$$

This can be written in matrix notation;

$$\begin{pmatrix} P^{(0)}_{N_1} \\ P^{(M)}_{N_2} \\ P^{(M+1)}_{N_3} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ N_1 & N_2 & N_3 \\ (-5)^{N_1} & (-5)^{N_2} & (-5)^{N_3} \end{pmatrix}$$

As before set $P^{(0)}_0 = 1, P^{(M)}_M = 1, P^{(M+1)}_{M+1} = 1$ and therefore $N_1 = 0, N_2 = M, N_3 = M+1$

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & M & M+1 \\ 1 & (-5)^{M} & (-5)^{M+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now find the inverse of the middle matrix;

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & M & M+1 \\ 1 & (-5)^{M} & (-5)^{M+1} \end{pmatrix}^{-1} =$$

$$\frac{1}{(-5)^M(-6M-1)+1} \begin{pmatrix} M(-5)^{M+1} - (M+1)(-5)^M & (-5)^M - (-5)^{M+1} & 1 \\ M+1 & (-5)^{M+1} - 1 & -M-1 \\ -M & 1-(-5)^M & M \end{pmatrix}$$

Now we have the values for x_{ij} so can find the values $P^{(k)}_N$

$$P^{(0)}_N = M(-5)^{M+1} - (M+1)(-5)^M + ((-5)^M - (-5)^{M+1})N + (-5)^N)/((-5)^M(-6M-1)+1)$$

$$P^{(M)}_N = (M+1+((-5)^{M+1}-1)N + (-M-1)(-5)^N)/((-5)^k(-6M-1)+1)$$

$$P_N^{(M+1)} = (-M + (1 - (-5)^M)N + M(-5)^N)/((-5)^M(-6M - 1) + 1)$$

These are all in terms of N and M and are completely generic for any starting and finishing point for this specific game.
So for example if I start with N=5 and end M=10;

$$\begin{aligned} P_5^{(0)} &= 0.5082 \\ P_5^{(10)} &= 0.4098 \\ P_5^{(11)} &= 0.0820 \end{aligned}$$

This concurs with the original example as you are more likely to ruin than to be successful and lose, however it is much closer in this example. Now i will look at whether these numbers change very much when you increase the size of the game. To be able to compare consistently I will keep the stopping strategy at double the initial stake i.e. M=2N.

Table 2.2:

N	k=0	k=2N	k=2N+1
5	0.508201967	0.409778351	0.082019681
10	0.504132231	0.413223158	0.082644611
15	0.502762431	0.414364641	0.082872928
20	0.502074689	0.414937759	0.082987552
50	0.500831947	0.415973378	0.083194676
100	0.50041632	0.416319734	0.083263947
150	0.500277624	0.416435314	0.083287063
200	0.500208247	0.416493128	0.083298626

This shows that as the amount of money you start with increases the chance of ruin decreases or the chance of success increases. This seems to be heading towards a limit;

$$P_N^{(0)} = M(-5)^M - (M+1)(-5)^M - ((-5)^M - (-5)^{M+1})N + (-5)^N)/((-5)^M(-6M-1))$$

$$P_N^{(0)} = 2N(-5)^{2N} - (2N+1)(-5)^{2N} - ((-5)^{2N} - (-5)^{2N+1})N + (-5)^N)/((-5)^{2N}(-12N-1))$$

Now find the limit as $N \to \infty$;

$$\Rightarrow \frac{2N - (2N+1) - (1 - (-5))N + (-5)^{-N})}{(-12N - 1)}$$

$$\Rightarrow \frac{-1 - 6N + \frac{1}{(-5)^N}}{(-12N - 1)}$$

$$\Rightarrow \frac{-1/6N - 1 + \frac{1}{6N(-5)^N}}{-2 - 1/6N}$$

$$as \quad N \to \infty \quad P_N^{(0)} \to \frac{0 - 1 + 0}{-2 - 0} = 0.5$$

This limit tends to 0.5, which agrees with the data.

From the examples above you can see that although a game which is intuitively fair, you have more chance of ruin than of success. This helps the house as seemingly fair games actually give the house an advantage. This can be used by casinos as it gives the casino the edge and allows them to make money in the long run.

2.2 Coin Tossing - Penney's Game

There exist certain betting games involving coin tossing similar to the example above. Intuitively the game seems fair, however with a small amount of prior knowledge and the correct betting strategy the game always give the house or person offering the game an advantage.

2.2.1 Penney's Game

The example I will use to explain this concept is a game called Penney's Game. This game has also been used by magicians in their tricks to unsuspecting members of the public to baffle them with their predictive power, including Derren Brown [6].

Penney's Game was invented by Walter Penney and is a coin tossing game between two players. The first player picks a combination of heads (H) and tails (T) that they think will come up. Then the second player picks his combination of heads and tails, making sure the length of the combinations are the same. The coin is then flipped continuously until a player completes their combination of heads and tails. You would instinctively assume that as each toss of the coin is independent and the coin is fair that each combination is as likely as any other i.e. HHH is as likely to come up as THH. However if the second player plays the correct strategy and picks their best combination they always have an advantage over the first player. The length of combination needs to be 3 or longer for this to be always correct.

Imagine the simplest non-trivial example of a combination of two. Player 2 cannot always guarantee a win, as with lengths of 3 or more, as it depends of what player 1 picks, however it is the simplest example to describe the concept. Imagine player 1 picks HH and player 2 picks TH. The four possible outcomes are HH, HT, TH and TT; all are equally likely outcomes. If HH or TH comes up then the game is over. If HT or TT comes up then if the third toss is a H, player 2 wins. If the third toss is a T, we are in the same position as the last two tosses were TT so a H on the fourth toss would let player 2 win and a T would continue the game. Therefore as soon as a T comes up, there is no possible way for player 1 to win. Therefore in this example the only way for player 1 to win is to get HH in the first two tosses, i.e probability of 1/4. This can be expanded to much larger combinations easily when player 1 picks a combination of only one type e.g. HHH or TTTTT. This is because of similar analysis from before that if player 1 picks all heads, then all player 2 needs to pick is THHH... and then as soon as a tail appears then player 1 cannot win. Therefore the probability of winning as player 1 is;

$$P_{winning} = \frac{1}{2^n} \qquad \text{with n=length of combination}$$

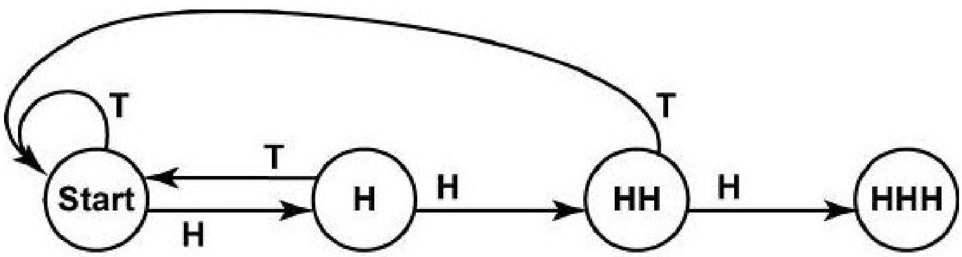
However the probabilities when not such an obvious poor choice is chosen are much closer. I will consider a game of length 3 as this is the smallest length at which certain choices from player 2 increase their probability of winning whatever player 1 choses. The best strategy for player 2 is to copy the first two choices of player 1 and have them as your last two choices, then put as your first choice the opposite of player 1's middle choice. All possible choices from player 1 and the combinations player 2 should choose and the odds in favour of player 2 winning are in the table below [9];

Table 2.3:

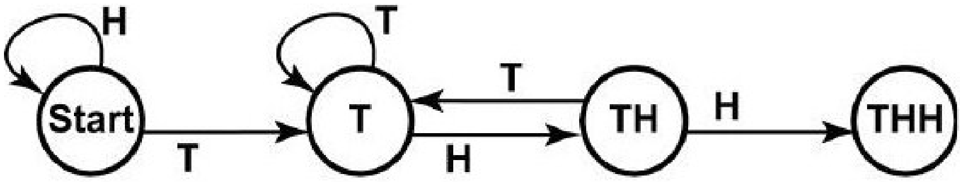
Player 1 choice	Player 2 choice	Odds in favour of player 2
HHH	THH	7-1
HHT	THH	3-1
HTH	HHT	2-1
HTT	HHT	2-1
THH	TTH	2-1
THT	TTH	2-1
TTH	HTT	3-1
TTT	HTT	7-1

The odds in favour of player 2 in the table above can be calculated by the use of markov chains.

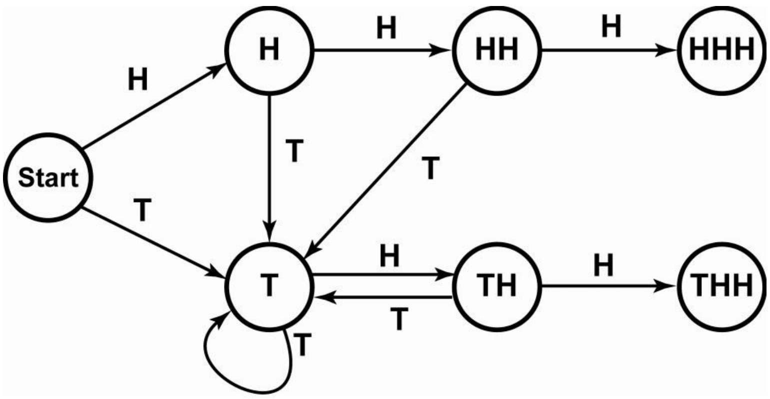
This can be seen in the following diagrams;



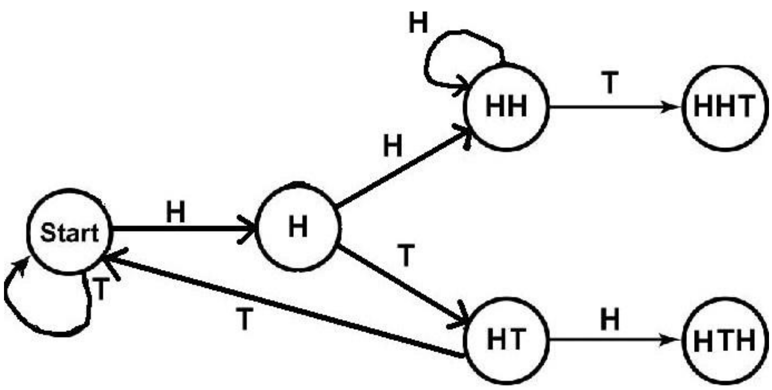
The image above shows that for a combination of HHH, as soon as a tail has been tossed, the game resets back to the start.



The image above shows that with THH, once a tail has been thrown it locks into the next step and the player can never return back to the start.



This markov chain shows the two combinations together and their race to win. It clearly shows that THH is much more likely to win. The only way for HHH to win is to get HHH in the first 3 tosses as there are no arcs going to HHH once a tail has been tossed. This shows that the probability of HHH beating THH is 1/8 or as seen in the table above the odds are 7 to 1 in favour of player 2. A slightly more complicated markov chain can be seen below for comparing HTH against HHT.



This shows that as soon as you get to HH then you are certain to reach HHT before HTH, but if you reach HT then you only have 50% chance of reaching HTH or starting the whole sequence again. As HH and HT are equally likely to appear then you are twice as likely to reach HHT before HTH. This is shown in table 2.3 with odds of player 2 winning being 2-1. Similar analysis can be done to achieve the other probabilities.

Penney's Game is an example of a situation that seems instinctively fair, but on closer inspection it is far from it. If player 2 has the right strategy, they can give themselves a great advantage. [7] [8]

Chapter 3

Kelly Criterion

John Larry Kelly Jnr (1923-1965) was an American scientist who worked on imperfect communication channels at AT&T Bell laboratory. His research concerned signal noise on long distance communication wires. While working with a colleague Claude Shannon who was working on information theory, Kelly produced a paper in March 1956 called 'A New Interpretation of Information Rate'. In the opening paragraph this paper states;

"If the input signals to a communication channel represent the outcomes of a chance event on which bets are available at odds consistent with their probabilities (i.e. "fair" odds), a gambler can use the knowledge given him by the received symbols to cause his money to grow exponentially." [5]

This means that with added information communicated over a channel before the results of an event allows the player to increase his chances of winning and exploit a positive expectation. The Kelly Criterion tells the gambler the fraction of capital to bet to maximise their expected growth, but also to minimise the probability of gamblers ruin. It is very important that both are successful as to only maximise the growth you would bet all of your capital and a single loss would bankrupt you.

The Kelly Criterion can only be used in the long run when you have a positive expectation on an event. This positive expectation is achieved by the extra information in the definition of Kelly Criterion. In practice most casino games are favoured towards the house and so Kelly Criterion cannot be used, however there are certain other strategies that be used alongside the Kelly Criterion to create this positive expectation. I will be exploring these in chapter 6.

3.1 Derivation

I will now move on to deriving the Kelly Criterion with the help of the following example.

Imagine a biased coin with $p > 0.5$ and $q = 1 - p$, hence $q < 0.5$. If your initial capital is C_0 and even odds are offered on the game, you double your money if you win and lose your stake if you lose. The player bets a proportion of his capital a at each trial. Therefore after one toss you either have;

$$C_1 = C_0 + C_0 a$$

or

$$C_1 = C_0 - C_0 a$$

depending on whether you win or lose. If the game is then repeated many times your capital after n trials is;

$$C_n = (1 + a)^{n-k}(1 - a)^k C_0 \tag{3.1}$$

with k being the number of losses.

You can then derive the Kelly Criterion from this by taking logarithms and then maximising the rate of growth.

$$\begin{aligned}
logC_n &= log((1 + a)^{n-k}(1 - a)^k C_0) \\
logC_n &= log(1 + a)^{n-k} + log(1 - a)^k + logC_0 \\
logC_n &= logC_0 + (n - k)log(1 + a) + klog(1 - a) \\
G_n &= \frac{1}{n}log\left(\frac{C_n}{C_0}\right) = \frac{n-k}{n}log(1 + a) + \frac{k}{n}log(1 - a)
\end{aligned}$$

where G_n is the growth rate of your capital. To maximise your capital you need to maximise the expected value of this rate of growth and find the fraction to bet a to achieve this

$$\begin{aligned}
E(G) = g(a) &= E\Big(\Big(\frac{n-k}{n}\Big)log(1 + a)\Big) + E\Big(\frac{k}{n}log(1 - a)\Big) \\
g(a) &= plog(1 + a) + qlog(1 - a)
\end{aligned}$$

by the Law of Large Numbers.

The aim is obviously to maximise this expected growth rate.

$$\begin{aligned}
g^{/}(a) &= \frac{p}{1 + a} - \frac{q}{1 - a} = \frac{p - q - a(p + q)}{(1 + a)(1 - a)} \\
0 &= \frac{p - q - a}{(1 + a)(1 - a)}
\end{aligned}$$

As the proportion of capital a is not 1 then the above is still defined;

$$p - q - a = 0$$

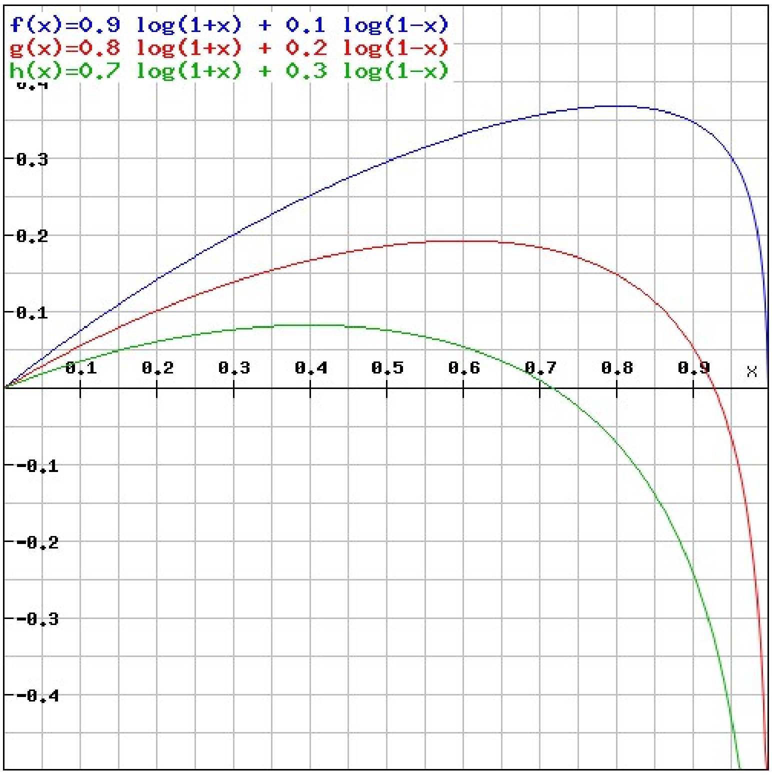
so the maximum expected growth is when

$$a^* = p - q$$

To check this is a maximum differentiate again;

$$\begin{aligned} g^{//}(a) &= -\frac{p}{(1+a)^2} - \frac{q}{(1-a)^2} \\ &< 0 \end{aligned}$$

Therefore this is a maximum.
This can also be seen in the following graph that I have plotted [18]



The x axis shows your stake as a fraction of capital and the y axis shows the rate of growth of capital

I have picked various different values for p in order to see how the graph changes. I have chosen $p = 0.9, 0.8, 0.7$ and the corresponding maximums 0.8, 0.6, 0.4 seen on the graph also concur with the Kelly Bet.

However this is specific to the example I have been explaining in the fact that the odds are even. If the odds are not even, but instead are b to 1 i.e. pays b profit on a unit stake then calculation changes slightly. I have followed the odds of winning through the calculation in the following;

$$C_1 = C_0 + C_0ab$$

or

$$C_1 = C_0 - C_0a$$

depending on whether you win or lose.

$$\begin{aligned}
C_n &= (1+ab)^{n-k}(1-a)^k C_0 \\
\Rightarrow logC_n &= logC_0 + (n-k)log(1+ab) + klog(1-a) \\
\Rightarrow G_n = \frac{1}{n}log\left(\frac{C_n}{C_0}\right) &= \frac{n-k}{n}log(1+ab) + \frac{k}{n}log(1-a) \\
\Rightarrow E(G) = g(a) &= E\Big(\Big(\frac{n-k}{n}\Big)log(1+ab)\Big) + E\Big(\frac{k}{n}log(1-a)\Big) \\
g(a) &= plog(1+ab) + qlog(1-a) \\
\Rightarrow g^{/}(a) &= \frac{pb}{1+ab} - \frac{q}{1-a} = \frac{pb-q-ab}{(1+ab)(1-a)} \\
0 &= pb - q - ab \\
\Rightarrow a^* &= \frac{pb-q}{b}
\end{aligned}$$

To check this is still a maximum differentiate again;

$$\begin{aligned}
g^{//}(a) &= -\frac{pb}{(1+ab)^2} - \frac{q}{(1-a)^2} \\
&< 0
\end{aligned}$$

Therefore the Kelly Bet for a favourable game is;

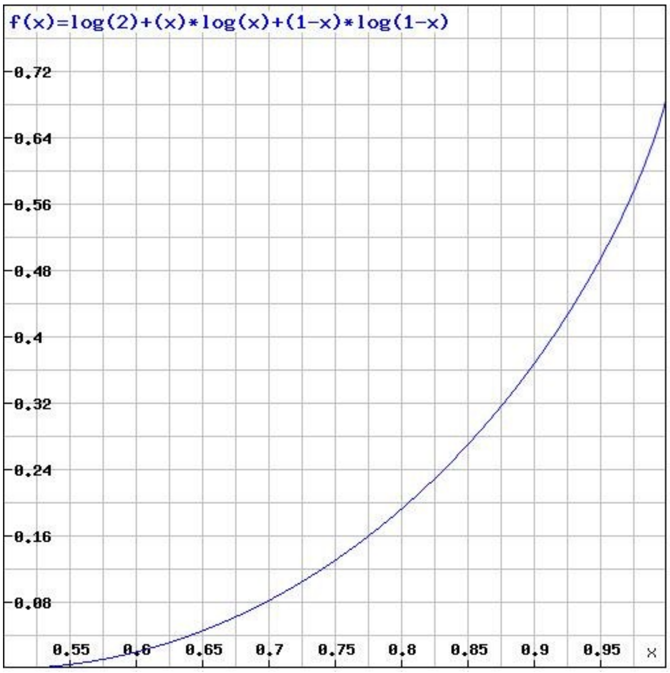
$$a^* = \frac{bp-q}{b}$$

3.2 Rate of Growth

The expected rate of growth can be calculated by using the fraction of capital from the Kelly Bet. If we say the optimum fraction to bet is a^* then for the original example with even odds;

$$\begin{aligned}
g(a^*) &= plog(1+a^*) + qlog(1-a^*) \\
&= plog(1+(p-(1-p))) + qlog(1-((1-q)-q) \\
&= plog(2p) + qlog(2q) \\
&= plog2 + plog(p) + qlog2 + qlog(q) \\
&= log2 + plog(p) + qlog(q)
\end{aligned}$$

Now if I plot this graph [18] you can clearly see that as soon as your probability of winning is above 0.5 then by using the Kelly Bet you can achieve a positive rate of growth. This rate of growth increases exponentially.



The x axis shows the probability of winning and the y axis shows the expected rate of growth of capital

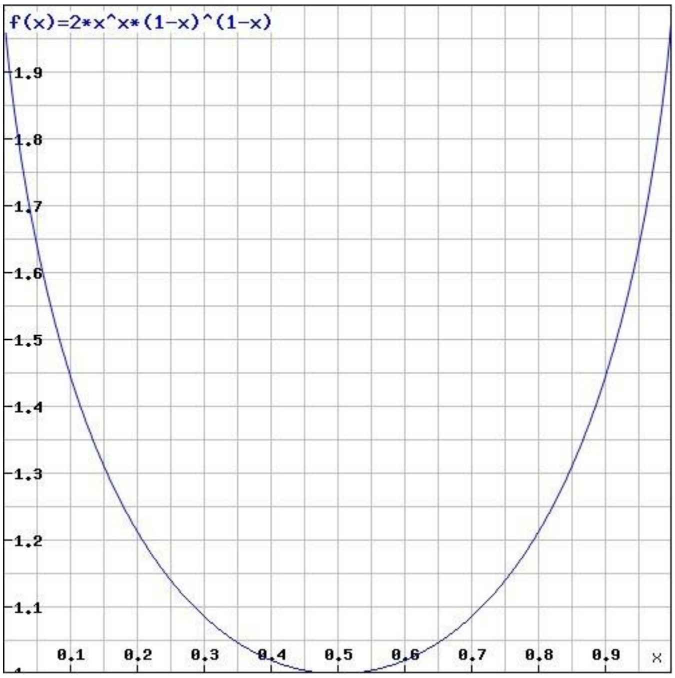
3.3 Deviation from the Kelly Bet

Deviations from the kelly Bet can have serious implications on your expected rate of growth. In this section I will be looking at overbetting and underbetting and how this effects your rate of growth of capital. Looking back at formula (3.1) at the total capital C_n, I can manipulate this formula to compare deviations in the Kelly Bet.

Starting with the original Kelly Bet a^*;

$$\begin{aligned} C_n &= C_0(1+a^*)^{n-k}(1-a^*)^k \\ &= C_0(2p)^{n-k}(2-2p)^k \\ &= C_0 2^n p^{n-k}(1-p)^k \\ &\approx C_0 2^n p^{np}(1-p)^{(1-p)n} \\ &= C_0(2p^p(1-p)^{(1-p)})^n \end{aligned}$$

Now as n$\rightarrow \infty$ and C_0 is a constant, then we want middle component to be as large as possible. If I set the middle component as function of p and call it F(p) then $F(p) = 2p^p(1-p)^{(1-p)}$. Now if I graph this function to see what happens for different probabilities;



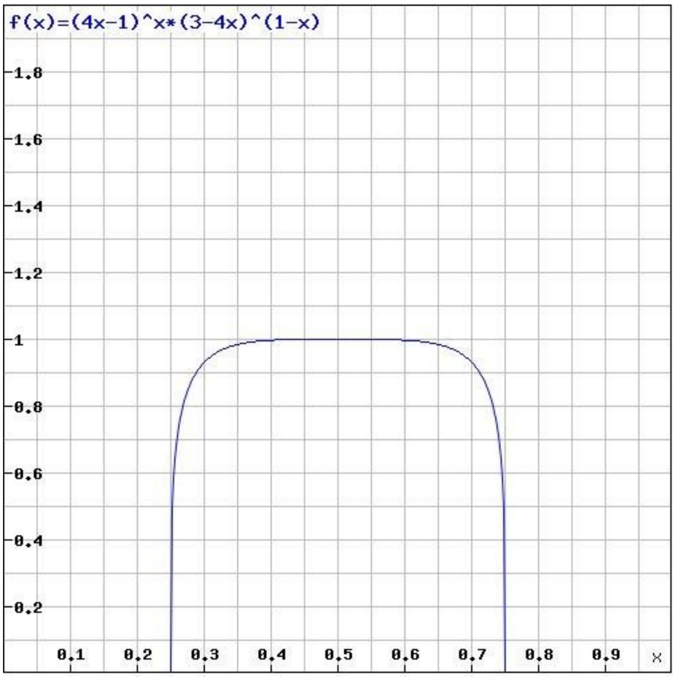
The x axis shows the values of p and the y axis shows the value of F(p)

You can clearly see that for all values of p>0.5 then F(p)>1. Therefore as n$\rightarrow \infty$ the capital $C_n \rightarrow \infty$. As this is for the actual Kelly Bet itself you would expect this to happen because of the positive rate of growth.

Now I will double the Kelly Bet to see what effect this has i.e. $2a^*$;

$$\begin{aligned} C_n &= C_0(1+2a^*)^{n-k}(1-2a^*)^k \\ &= C_0(4p-1)^{n-k}(3-4p)^k \\ &\approx C_0(4p-1)^{np}(3-4p)^{(1-p)n} \\ &= C_0((4p-1)^p(3-4p)^{(1-p)})^n \end{aligned}$$

Similar to before I will graph the middle component;



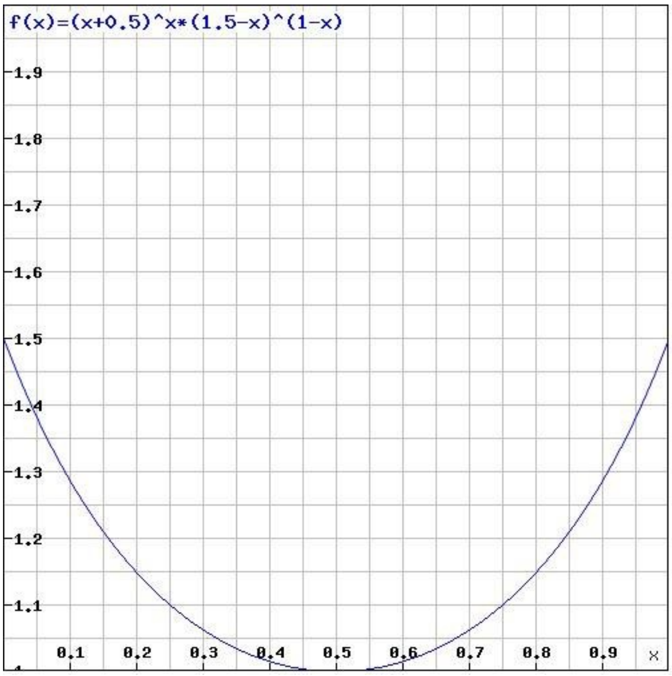
The x axis shows the values of p and the y axis shows the value of F(p)

From this graph it is clear to see that for all values of p, the value of F(p)≤ 1. Therefore as n$\rightarrow \infty$ $C_n \rightarrow 0$. In the real world this means that if you bet double the Kelly Bet in the long run you will reach gamblers ruin and become bankrupt.

Now I will halve the Kelly Bet to see what effect this has i.e. $0.5a^*$;

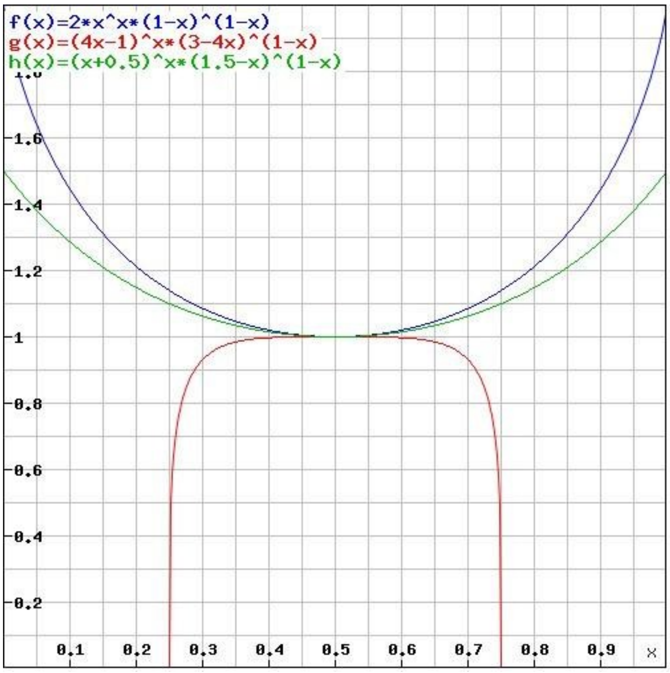
$$\begin{aligned} C_n &= C_0(1+0.5a^*)^{n-k}(1-0.5a^*)^k \\ &= C_0(p+0.5)^{n-k}(1.5-p)^k \\ &\approx C_0(p+0.5)^{np}(1.5-p)^{(1-p)n} \\ &= C_0((p+0.5)^p(1.5-p)^{(1-p)})^n \end{aligned}$$

Similar to before I will graph the middle component;

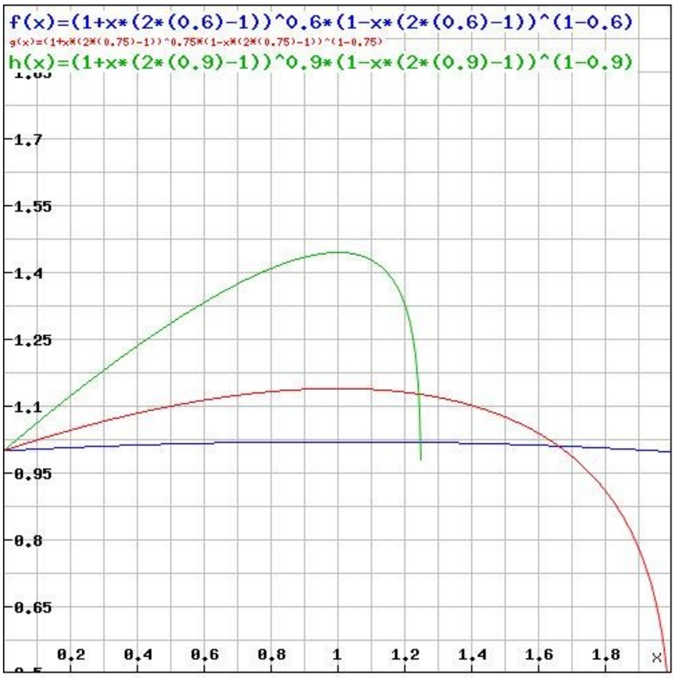


The x axis shows the values of p and the y axis shows the value of F(p)

This is much more similar to betting the actual Kelly Bet, however it produces slightly lower values of F(p) for each p. It also has F(p)>1 for all p, hence as n$\rightarrow\ \infty$ $C_n\ \rightarrow\ \infty$ and in the long run your capital will exponentially grow. All three examples can be compared in the following graph;



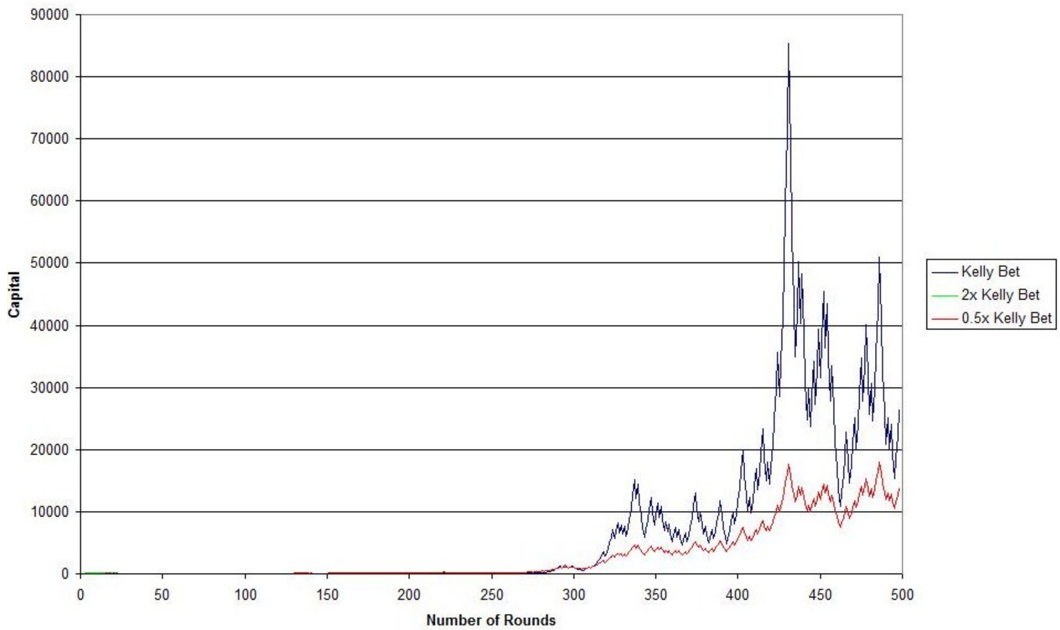
This shows that the expected rate of growth varies a lot for different multiples of the Kelly Bet. I will now plot the function of different multiples of the Kelly Bet with a fixed probability, to show that the maximum is in fact 1 times the Kelly Bet as expected.



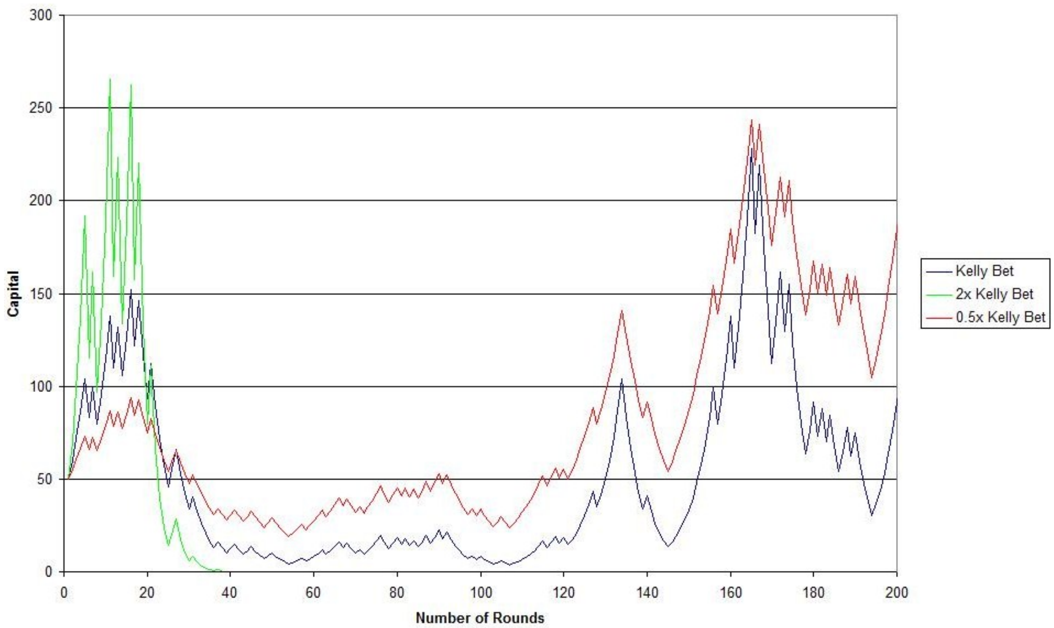
The x axis shows the proportion of the Kelly Bet to bet and the y axis is proportional to the rate of growth of capital

The probabilities I have chosen are $p = 0.6$ in blue, $p = 0.75$ in red and $p = 0.9$ in green. This shows that as expected the maximum rate of growth is at 1 times the Kelly Bet. It also shows that betting less than the Kelly Bet reduces the rate of growth, but still keeps it positive. However if you bet a lot more than the Kelly Bet, the rate of growth can become negative and in the long run your capital will tend to 0, seen in the graph when the value drops below 1.

I will now plot a simulation of the same game, comparing different multiples of the Kelly Bet. I have chosen to use 500 trials as the Kelly Criterion is effective only in the long run, the probability of success is 0.6 and the starting capital is £50.



It is not very clear what is happening in the first 200 trials because of the scale of the graph so I have decided to zoom in on the first 200 trials.



This is a specific example of this game, other simulations would produce different graphs. In this example betting 2x Kelly Bet causes the player to ruin after only 40 trials. After 500 trials betting the Kelly Bet itself produces the most capital as you would expect however in the last 100 trials the capital is very variable. The capital varies by tens of thousands of pounds in each bet when only starting with £50. When using the half Kelly strategy the volatility of capital is much less. It is a far more stable strategy when playing for many repeated trials. After the first 200 trials the half Kelly Bet produces the most capital and so it is only in the last 200 trials when the Kelly Bet really starts to dominate.

This may imply that betting for a large number of repeated games anything other than the Kelly Bet is futile. However if you bet slightly less than the Kelly Bet, you will still see a positive rate of growth, but the results are a lot less volatile and the chance of ruin and reaching zero capital in the short run is also a lot less. Also gamblers tend to overestimate their probability of winning and so underbetting can help to balance this out. This strategy is used throughout the gambling world especially with beginners to the Kelly Criterion, who are fearful of losing all their money.

Chapter 4

Progressive Betting Systems

I will now be looking into other betting strategies apart from the Kelly Criterion. Many of these focus positive or negative progression systems or try to hedge their bets for a small loss or large profit. The following strategies can be used in games with an even payoff and near even probabilities of winning i.e. tossing a coin, red/black on roulette.

Progressive betting strategies are based on previous outcomes of the game and try to increase profit through winning streaks. These strategies usually aim to reduce the number of paths to losses and increase the number to a win, however within the reduced number of losses they tend to be much greater losses, whereas the wins tend to be smaller as well. Progressive strategies usually have a number of levels of bets that you progress though after different outcomes of a bet. A positive strategy means you would progress after a win and a negative strategy would progress after a loss. For example a five-level positive progressive strategy like 1-2-3-5-1 would mean that you bet 1 unit until you win then bet 2 units until you win. You keep moving through the progression until you win betting 5 units then start again betting 1 unit. Some progressive strategies also move backwards and forwards though the progression depending on whether you win or lose.

Unlike the Kelly Criterion these strategies generally will only work in the short run, as in the long run the casino with their infinite bankroll and odds in their favour will eventually reduce you to bankruptcy. However this can be good thing as it can make a profit in a very short amount of time.

4.1 Martingale

The Martingale strategy is one of the simplest strategies that can be played and can be considered a negative progressive strategy. In a game with even payout, you place a bet and if you lose you double the bet. If you win you restart with your minimum bet. Let γ_i be the stake for the i^{th} game.

$$\gamma_i = \begin{cases} 2\gamma_{i-1} & \text{if a loss on the } i-1^{th} \text{ game} \\ \gamma_1 & \text{if a win on the } i-1^{th} \text{ game} \end{cases}$$

For example tossing a coin or playing roulette and betting on red, every time you lose you double your stake until you win then restart. Theoretically in the long run you cannot lose and your expected growth would be infinite. However although Martingale in one of the most widely known and implemented strategies, its chance of gamblers ruin is also extremely high. This is because the player needs to have an infinite wealth for the strategy to be successful. Otherwise a long run of tails or black in roulette would bankrupt the player. As the size of bets increases rapidly 2^nx(minimum bet) then a losing streak will cause you to ruin. In practice no player has infinite wealth or if they did they wouldn't be gambling to make money.

If a player starts with £1000 and bets £1 as minimum bet, then a losing streak of 10 will bankrupt them, but a winning streak of 10 will only cause a profit of £10. However if your bankroll is large enough a profit can be achieved, although the overall profit for each win will only be the minimum bet. i.e.

Table 4.1:

Stake	Win/Loss	Profit
1	L	-1
2	L	-3
4	L	-7
8	L	-15
16	L	-31
32	L	-63
64	L	-127
128	W	1

The Martingale strategy is one of the oldest strategies and could be used effectively in the past. However in modern times all casinos have betting limits that make the use of Martingale less effective. It is also clear from the example above that there is an obvious point to stop when you win. This can be brought in as part of the strategy to ensure you always leave with a profit.

4.2 Anti-Martingale

The Anti-Martingale strategy is very similar to the Martingale strategy. Instead of trying to cover losing streaks as with the Martingale, it aims to maximise profit from winning streaks. It can be considered a positive progressive strategy as it states you should double the stake when you win rather than when you lose. In this case γ_i becomes

$$\gamma_i = \begin{cases} \gamma_1 & \text{if a loss on the } i-1^{th} \text{ game} \\ 2\gamma_{i-1} & \text{if a win on the } i-1^{th} \text{ game} \end{cases}$$

The Anti-Martingale can make big profits when on a winning streak, however it is key to stop playing the game before a loss. Otherwise all the profit you have made will be wiped out. However unlike the Martingale strategy where you are in negative profit until you finally win and make a slight profit, in Anti-Martingale you are in profit until you lose then go back to break even.

Table 4.2:

Stake	Win/Loss	Profit
1	W	2
2	W	4
4	W	8
8	W	16
16	W	32
32	W	64
64	L	0

The Anti-Martingale betting strategy is also called the Paroli progressive betting strategy. [14]

4.3 D'Alembert

The D'Alembert betting system is also known as Pyramid betting system. It is called D'Alembert strategy after the French Mathematician Jean le Rond D'Alembert whose Law of Equilibrium it is based around. The Law of Equilibrium implies a balance of outcomes in the long run. This can be exploited in this strategy by the fact that numbers of outcomes should eventually be equal. Imagine a game with even payout and then you increase your bet by 1 unit every time you lose and decrease it by 1 unit every time you win. This can be written in γ_i notation as;

$$\gamma_i = \begin{cases} \gamma_{i-1} + 1 & \text{if a loss on the } i-1^{th} \text{ game} \\ \gamma_{i-1} - 1 & \text{if a win on the } i-1^{th} \text{ game} \end{cases}$$

In the following example it is clear to see how the D'Alembert strategy works in practice.

Table 4.3:

Stake	Win/Loss	Profit
1	L	-1
2	L	-3
3	L	-6
4	L	-10
5	W	-5
4	W	-1
3	W	2

From this example you can see that you have lost 4 times and won only 3 times but still in profit. In this strategy there is no obvious fixed point at which to stop unlike Martingale. Therefore it is very important to decide on one before you start as in the long run as with all progressive strategies in casinos no profit will be achieved. Looking at the combinations of wins and losses the D'Alembert strategy makes a lot of paths to small wins and few paths to large losses. [15]

4.4 Contra-D'Alembert

The Contra-D'Alembert strategy is basically the opposite to D'Alembert in the way Martingale and Anti-Martingale are opposites. Contra-D'Alembert states that you increase your bet by 1 unit every time you win and decrease your stake by 1 unit every time you lose.

$$\gamma_i = \begin{cases} \gamma_{i-1} + 1 & \text{if a win on the } i - 1^{th} \text{ game} \\ \gamma_{i-1} - 1 & \text{if a loss on the } i - 1^{th} \text{ game} \end{cases}$$

The following example clearly explains how the strategy works in practice

Table 4.4:

Stake	Win/Loss	Profit
1	W	1
2	L	-1
1	W	0
2	W	2
3	L	-1
2	W	1

The Contra-D'Alembert strategy, in the opposite way to D'Alembert strategy produces a lot of paths to small losses and few paths to large wins. It similarly has no obvious stopping strategy associated with it and so is important to decide on one before you start.

4.5 Labouchere

The Labouchere system is primarily used in roulette and is often called the cancellation system or split martingale. The player decides at the start how much profit he wants from his capital and then writes down a sequence of numbers that equal the profit required. I will call the sequence of numbers $s_1, s_2, s_3...s_n$ then the profit required S. Therefore;

$$S = s_1 + s_2 + s_3 + ... + s_n$$

Now the stake for each round of betting is;

$$\gamma = s_1 + s_n$$

However if you win the round you remove s_1 and s_n from the list and start again with a reduced list. If you lose the round then $s_1 + s_n \rightarrow s_{n+1}$ and is added to the end of the list. As the game continues when you win you remove 2 numbers and if you lose you only add one number, then equal number of wins and losses will still produce a profit. This is because as long as you complete the list and run out of numbers then you will make your profit S. This is shown in the following example where S=25 and the sequence is 3,5,4,7,5,1;

Table 4.5:

Sequence	Stake	Win/Lose	Profit
3,5,4,7,5,1	4	L	-4
3,5,4,7,5,1,4	7	L	-11
3,5,4,7,5,1,4,7	10	W	-1
5,4,7,5,1,4	9	W	8
4,7,5,1	5	L	3
4,7,5,1,5	9	W	12
7,5,1	8	L	4
7,5,1,8	15	L	-11
7,5,1,8,15	22	W	11
5,1,8	13	W	24
1	1	W	25

The theory behind this system comes from the fact that for a game with 50/50 chance of winning the number of wins or losses will be roughly equal. As the list shortens by 2 for a win and lengthens by only 1 for a loss, it is possible for you to complete the list if you win more than 1/3 of the time. For example a game of length 7 and you win 5 times and lose 3 times then your win percentage is 62.5% and you win the profit you desire. However if you win 43,600 times and lose 87,193 times you still complete the game and win the desired profit with a win percentage of only 33.3%. [16]

This strategy has been widely used in roulette when betting on an event with even payoff i.e. red/black, even/odd, first 18/last 18. However with roulette the odds of winning these are not 50/50 because of the 0 in which case the house wins. Therefore the chance of coming up red is 18/37. The Labouchere strategy helps bring these odds from the favour of the house to the player.

4.6 Reverse Labouchere

The Labouchere system can reversed so that when you win you add you stake onto the end of your sequence and when you lose you remove the first and last numbers in your sequence. This is known as the Reverse Labouchere. Unlike the Labouchere where you state how much a player wants to win, the Reverse Labouchere states the most a player is willing to lose. Players usually continue until the table limit is reached or until a personal limit is achieved.

4.7 Oscar's Grind

Oscar's Grind is extremely successful in maximising the number of small wins in relation to large losses. Oscar's Grind is designed to make only 1 unit profit and the strategy actually prevents you making more. You start by betting 1 unit, if you win you stop with 1 unit profit. If you lose keep your stake the same and carry on betting. If you subsequently win add 1 to your stake and carry on betting unless this would cause you profit to become more than 1 unit. [13]

Chapter 5

Comparisons

Already in this report I have explained many of the strategies used in modern casino games. In this chapter I will discussing the good and bad points of some of these strategies and comparing them in order to determine which strategy could be considered best in certain situations.

5.1 Advantages and Disadvantages of Kelly Criterion

I have already explained in chapter 3 that the Kelly Criterion is usually considered the optimal strategy because it aims to maximise expected growth as well as minimising the chance of gambler's ruin. Many of the advantages of Kelly Criterion have already been mentioned. However we are still playing a game of chance and hence success can never be assured. Losses will be inevitable and the ease to which we can recuperate these losses also effects the success of a strategy. However with the Kelly Criterion it can become become very difficult to recuperate losses. When using the Kelly Criterion with a high probability of winning, it states that you should be betting a large fraction of your capital. For example p=0.9 with even odds then $a^* = 80\%$ of your capital. Even after being very successful and increasing your wealth, a couple of losses can cause you to lose all your winnings and more. In this example if you lost 80% of your capital then in the next trial you would need to increase you wealth 400% just to break even. This shows that once heavy losses occur it can be very difficult to get back into profit. This also works the other way; if you win 10% profit in 1 trial then in the next trial you only need to lose 9.09% to break even. This can be shown more clearly in the following;

If the loss from trial 1 is 100(1-d)% with 0< d <1 then;

$$
\begin{aligned}
C_n &= C_0(1+a)^{n-k}(1-a)^k \\
C_{n+1} &= C_0(1+a)^{n-k}(1-a)^{k+1} \\
&= C_0(1+a)^{n-k}(1-a)(1-a)^k \\
&= (1-a)C_n \\
&= dC_n
\end{aligned}
$$

Now to recuperate losses;

$$
\begin{aligned}
C_n &= C_{n+2} = \beta C_{n+1} = \beta d C_n \\
\Rightarrow \beta &= \frac{1}{d}
\end{aligned}
$$

Where β is the fraction of capital to bet on the second trial to break even. Therefore for the example above when you lose 80% of your capital $d = 0.2$ so $\beta = 5$ so $100(\beta - 1) = 400\%$. This is clearly a major flaw as you cannot have 400% of your capital to bet by the definition of capital in chapter 1. It is important to also add that by the definition of Kelly Criterion that you are betting a fraction of your total capital you can never actually become bankrupt, even if you lose every single trial. However by limitations of casinos with a minimum bet once this minimum is reached you will reach gambler's ruin if you lose as you will be betting 100% of you capital. Also limitations in the denominations of money can be a factor if there is no minimum bet as you cannot bet 40% of 1 penny.

Another major disadvantage of using the Kelly Criterion is that it can be very difficult to implement if the probabilities are not known. In this case the player will be estimating them from as much information as they have. Inaccuracies in the estimates are usually due to the fact that the majority of players tend to overestimate the edge they have over the house. However as seen above that it is very difficult to regain losses, therefore small inaccuracies in the estimates for the probabilities can be hugely detrimental. Some players may actually make their estimates then reduce the edge they think they have to try to compensate for any inaccuracies. This is because in chapter 3 I showed it was far more detrimental to long run success to overbet than to underbet.

Also the Kelly Criterion is only successful in the long run after many hundreds of trials. It can be very volatile because of the large fractions of capital betting hence a large bankroll is needed for success.

5.2 Kelly Criterion vs Progression Systems

The Kelly Criterion and Progressive Betting Strategies although aim to have the same eventual outcome, vary very differently. As mentioned before for the Kelly Criterion to be used a positive expectation is required. The odds of winning need to be in favour of the player and this cannot always be assured. Many of the other betting strategies do not need this for them to be successful. The Kelly Criterion also uses a lot more information than many of the progressive strategies as it includes the odds of winning and the probability of success on that single trial. This allows the game to vary throughout different rounds of betting and the Kelly Criterion accounts for this variation, whereas progressive strategies do not take this into account. The information that progressive strategies take into account are the results of previous outcomes. Increasing the bet for the current round will try to regain capital after a loss. However it is for this reason that a long losing streak can ruin a player. [10]

In table below it shows the number of losses required to bankrupt, or cause the bet to become below the minimum bet of £1, using different strategies. The probability of winning is $p = 0.7$ with even odds and minimum bet £1, capital £100. As the minimum bet is £1 I have rounded the stake bet on each round to the nearest whole number. I have also decided for this example that every trial is a loss to see how this affects the speed of your bankruptcy. This will help to show how detrimental the effects of a losing streak can be.

Table 5.1: Kelly Criterion

Round Number	Capital	Stake
1	100	40
2	60	24
3	36	14
4	22	9
5	13	5
6	8	3
7	5	2
8	3	1
9	2	1
10	1	1
11	0	N/A

Table 5.2: Martingale

Round Number	Capital	Stake
1	100	1
2	99	2
3	97	4
4	93	8
5	85	16
6	69	32
7	37	37 (64)
8	0	N/A

Table 5.3: Anti-Martingale/Contra-D'Alembert/Oscar's Grind

Round Number	Capital	Stake
1	100	1
2	99	1
3	98	1
4	97	1
5	96	1
...	...	...
98	5	1
99	2	1
100	1	1
101	0	N/A

Table 5.4: D'Alembert

Round Number	Capital	Stake
1	100	1
2	99	2
3	97	3
4	94	4
5	90	5
6	85	6
7	79	7
8	72	8
9	64	9
10	55	10
11	45	11
12	34	12
13	22	13
14	9	9(14)
15	0	N/A

In these tables Anti-Martinglae, Contra'D'Alembert and Oscar's Grind are all the same as they are defined to keep the bet at £1 until a win. To complete the tables for Labouchere and Reverse Labouchere the sequence you choose is vital to the rate at which you reach gambler's ruin. Whatever the length of sequence chosen a continual streak of losses means the game continues, and the numbers in the sequence determine the rate at which you will reach zero capital. The following tables compare a sequence of 2,2,2,2,2 and 5,5,5,5,5.

Table 5.5: Labouchere 2,2,2,2,2

Round Number	Capital	Stake
1	100	4
2	96	6
3	90	8
4	82	10
5	72	12
6	60	14
7	46	16
8	30	18
9	12	12 (20)

Table 5.6: Labouchere 5,5,5,5,5

Round Number	Capital	Stake
1	100	10
2	90	15
3	75	20
4	55	25
5	30	30
6	0	N/A

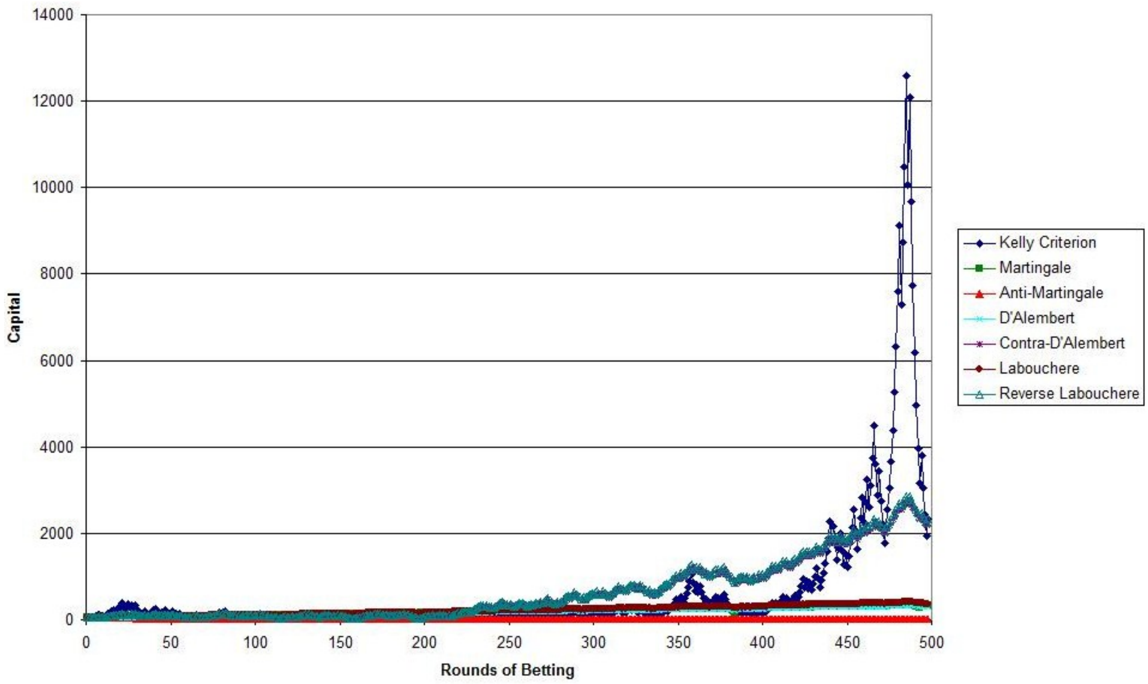
These tables comparing all the different strategies represent the shortest time to reach gambler's ruin. As you can clearly see Labouchere and Martingale cause ruin the fastest and Anti-Martingale, Contra-D'Alembert and Oscar's Grind all only reach gambler's ruin after 100 losses.

5.3 Graphical Representation

It will be very useful next to look at how easy it is to make profit using different strategies by setting up a simulation of a game and representing the growth of capital graphically. This game is defined by the following rules;

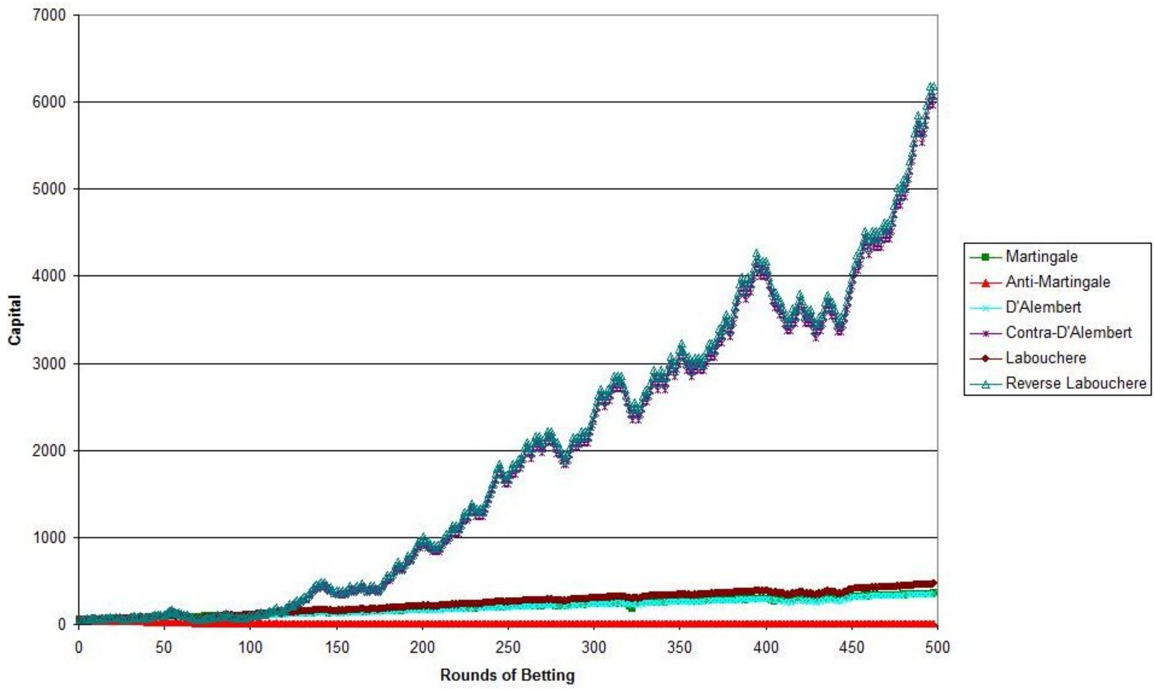
- There are only 2 outcomes a win or loss.
- The probability of winning is 0.6 and hence the probability of losing is 0.4.
- The game is repeated 500 times.
- Minimum bet is £1.
- The Kelly Bet is 40% of capital.
- Martingale, Anti-Martingale, D'Alembert and Contra-D'Alembert start at £1.
- Labouchere and Reverse Labouchere use a continuous sequence of 1's long enough so that it is not completed within 500 trials.

The following graph is a representation of this game. Every simulation of the game gives a different result, so each graph is a different example.



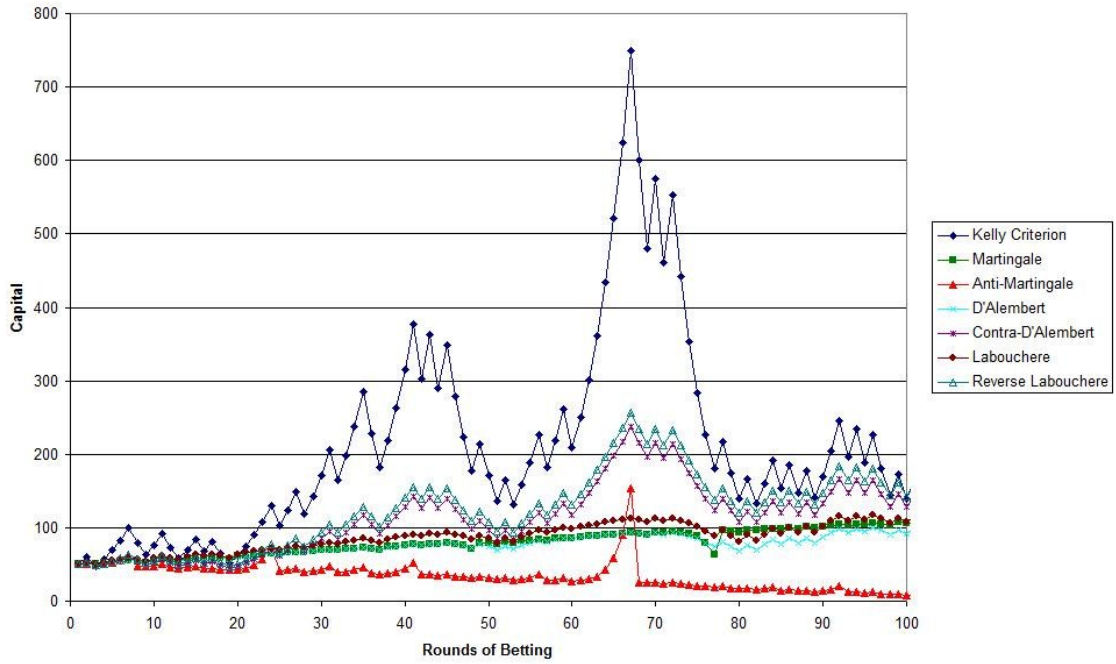
This graph shows that the Kelly Criterion achieves the most capital after 500 trials. However the Kelly Criterion growth is very volatile especially in the last 50 trials where a string of 5 losses lose you over £10000. Reverse Labouchere and Contra-D'Alembert produce a stable growth and ends on similar capital to using the Kelly Criterion.

In the next graph I have omitted the use of the Kelly Criterion to see more clearly the differences between the progressive strategies.

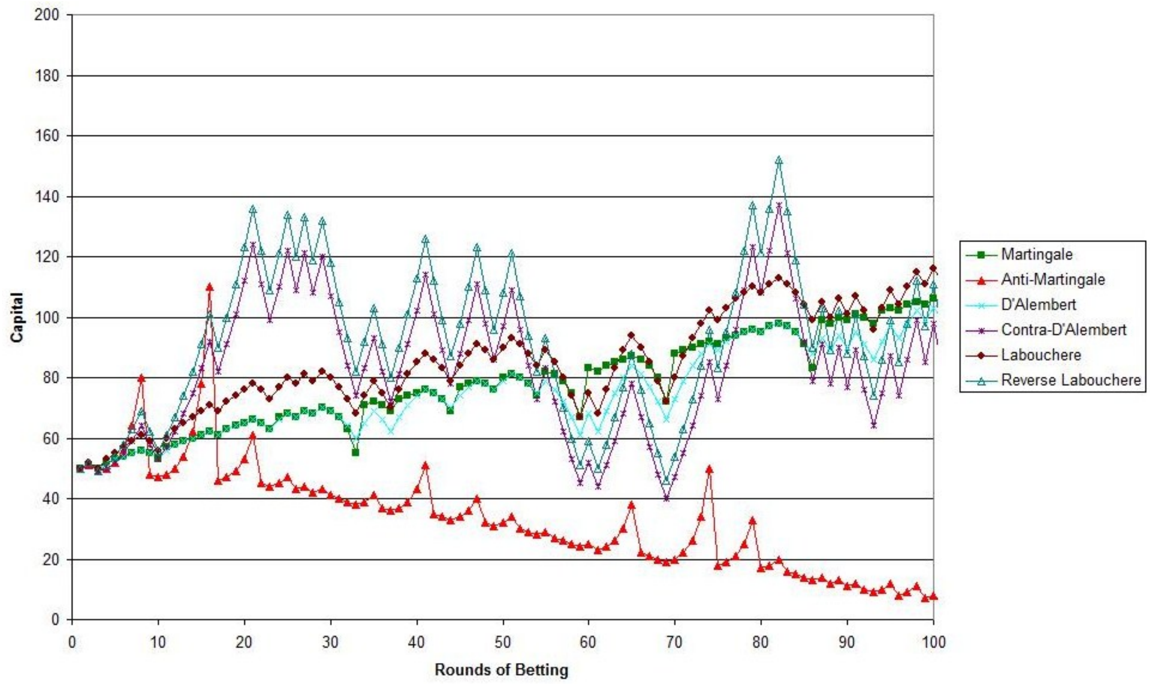


As before the Reverse Labouchere and Contra-D'Alembert produce the fastest growth from the progressive strategies. However from this graph you can see that the growth is reasonably volatile as well compared to Martingale, D'Alembert and Labouchere. The Anti-Martingale strategy has reached gamblers ruin after about 100 trials so this can be classed as the least successful.

The next graph I have reduced the number of trials to 100, to show how the strategies vary in the short term.



In this final graph I have reduced the number of trials to 100 and removed the Kelly Criterion to compare only the progressive strategies.



In the graphs of only 100 trials, it is unusual that the Kelly Criterion still produces the highest capital as it is only defined to be successful in the long run. It is also interesting to see that the Martingale strategy seems to produce a very stable and constant increase, although it takes the shortest losing streak to reach gambler's ruin.

Chapter 6

Applications - The Story of Edward Thorp

In this chapter I will be introducing the some of the strategies that can be used alongside the Kelly Criterion and how they can be utilised in casinos. I will also be discussing Edward Thorp and how he used this Criterion to make money.

The main problem with the Kelly Criterion is that a positive expectation is needed. The probability of winning needs to be in favour of the player, however all casino games are always in the favour of house. If this was not always true then the casinos would not offer the game. There are many other playing strategies that I have not already discussed in this report that help to actually change the probability of winning. The most notable strategies are card counting in blackjack and trajectory tracking in roulette.

Edward Oakley Thorp (1932-present) is an American mathematics professor specialising in probability theory. Although card counting had been explored before him, Thorp was the first man to prove that card counting can mathematically increase the probability of winning. He spent much of his time using the Kelly Criterion and variations of card counting in experiments at various casinos across the world to turn mathematical theory into millions of pounds.

6.1 Roulette

It is very easy to see that that the probability of winning roulette is stacked towards the house. Although there are 36 numbers to bet on, each paying 35-1 there is also a 0 so the chance of a single number coming up is actually 1/37. On American roulette wheels the odds favour the house even more as there is 00 as well, making the probability of a single number 1/38, but only paying 35-1. For examples in this chapter I will be using American roulette tables. Albert Einstein started looking at roulette tables and whether they were beatable, he believed them to always favour the house and is famously quoted saying "*No-one can possibly win at roulette unless he steals money from the croupier when he isn't looking.*" [12]

Thorp first began to look into roulette during his second year of graduate physics at UCLA. He wondered whether there was some mathematical way to predict the outcomes of spins, known as trajectory tracking. With old roulette tables the spins were never truly random as they were hand made and were slightly biased towards some numbers rather than others. However when Thorp began investigating the roulette wheel, in 1955, they were mechanically made and well maintained so they would spin true. It occurred to Thorp that with these better roulette tables the ball spinning round the table was akin to planets orbiting in our solar system and hence could be modeled and predicted.

Thorp realised that what is needed is the position of the ball and its velocity around the track to predict where the ball will land. For this to be successful, bets need to be placed after the ball is released and must be computed very quickly and easily. It is common in roulette for bets to be placed when the ball is in motion up until the croupier calls 'no more bets'. Thorp moved to MIT and teamed up with Claude Shannon, who he spent many years working with, developing systems for roulette and blackjack. They spent the next few years creating experiments with old roulette wheels and with trips to casinos to test their findings.

They started by dividing the motion of the ball into different segments and analyse them separately. The first section is the horizontal circular motion of the ball spinning around about the stationary part of the wheel (stator). It continues to spin until it slows down enough to fall down the sloped wall towards the centre. This assumes that they wheel is perfectly horizontal and the velocity of the ball depends of the number of revolutions before falling down. This means that from measuring the time for 1 revolution of the ball you could tell when the ball would start to fall towards the centre.

The second section is the ball falling down the sloped wall while still rotating around the stationary part of the wheel. It seems plausible that this would take the same amount of time for each trial as the ball starts to fall at the same speed and the wheel is assumed to be horizontal. In reality the wheel may not be horizontal but if this realised and taken into account it can become a great advantage to the player. Also in this section of the wheel there are obstacles that cause the ball to change its path. It was suggested by Thorp [11] that half the times when hitting these obstacles it makes a significant difference either stretching out the flight path causing it to rest further than expected or knocking it straight down causing it to rest prematurely.

The final phase is when the ball enters the spinning rotor at the centre of the wheel. This usually spins in the opposite direction to the ball to create a higher relative velocity and more of an error. When the ball enters this region there are a few situations that can occur. The ball can hit the edge of a pocket, bouncing the ball in the air, causing it to land further than anticipated or making it bounce backwards a few spaces. It can also kick it back out to the stationary part of the wheel for some revolutions before coming back. This spattering effect, with the ball bouncing around in an unpredictable way, increases as the velocity of the rotor increases and hence as the relative velocity of the ball increases. [11]

6.1.1 Errors

The errors in the predictability can be attributed to the following with the average amount of error attributable to each specific error written in brackets [11]:

- E1. Rotor timing (1.4 spaces)
- E2. Ball timing (5.5 spaces)
- E3. Variation in ball paths on rotor (unknown so will call it X)
- E4. Ball path down the stator, including error from obstacles (7)
- E5. Spattering effect when ball reaches the rotor (6)
- E6. Tilted wheel (unknown as specific to each table)

I will assume that the errors follow a normal probability distribution. Therefore the sum of more than 1 of these errors is the square root of the sum of the squares of the different errors. For example if you have errors 1 and 2 then the total error is:

$$E1 + E2 = \sqrt{1.4^2 + 5.5^2} = 5.68$$

The total error for all the error is therefore:

$$TotalError = \sqrt{1.4^2 + 5.5^2 + 7^2 + 6^2} = 10.8$$

This hasn't included E3, so it is obviously important to keep X as small as possible. This value X is very hard to predict as it varies greatly depending on the table and the croupier. Obviously the more the error the less the advantage. The following table shows the results of a bet on the best possible space and octant for different errors. The octant is the best space and the two either side of it.

Typical Error E (No. of Pockets)	Percent Advantage Betting on Best		Typical Error E (No. of Pockets)	Percent Advantage Betting on Best	
	Pocket	Octant		Pocket	Octant
0	3500.00	620.00	16	0.46	0.30
1	1278.53	611.06	17	− 1.62	− 1.72
2	610.69	467.86	18	− 3.01	− 3.07
3	376.52	328.65	19	− 3.90	− 3.94
4	258.12	236.98	20	− 4.46	− 4.49
5	186.76	175.71	21	− 4.81	− 4.82
6	139.09	132.62	22	− 5.01	− 5.02
7	105.00	100.89	23	− 5.13	− 5.13
8	79.41	76.65	24	− 5.19	− 5.19
9	59.54	57.60	25	− 5.23	− 5.23
10	43.77	42.38	26	− 5.24	− 5.25
11	31.19	30.18	27	− 5.25	− 5.25
12	21.24	20.52	28	− 5.26	− 5.26
13	13.54	13.03	29	− 5.26	− 5.26
14	7.73	7.37	30	− 5.26	− 5.26
15	3.47	3.24	∞	− 5.26	− 5.26

Figure 6.1: Table from [11]

This table shows that when the error is normally distributed the forecast error must be 16 spaces or less. Any more than 16, then the advantage goes back to the house. 16 spaces is 16/38 = 0.42 of a revolution of the wheel.

This unpredictability in the final stage of trajectory tracking makes it impossible to accurately predict the exact space where the ball will finish. Therefore Thorp and Shannon usually bet on the octant they thought it would land in. To predict where the ball will land is very difficult to calculate in the few seconds you have after the ball is launched until 'no more bets' is called. The calculations needed to be done on site and so a computer was needed and had to be hidden so the casinos did not realise what they were doing. Therefore they invented the first wearable computer. This was roughly the size of a cigartte packet and could be operated without casting suspicion. They tested it in the summer of 1961, with Shannon usually operating the computer and Thorp placing the bets acting as if they didn't know each other. Their wives also helped to keep look out in case the casino ever suspected anything. [9]

6.2 Blackjack

I will not be describing the detailed rules of blackjack as this would take too long and there are many varieties but the basic rules are as follows:

The aim of the game is to reach 21 without going over 21. Cards are at face value and ace can be either 1 or 11. If 21 is reached in 2 cards this is called blackjack and is the best hand in the game. Each player places a bet at the start of the round before any cards are dealt. Each player plays individually against the dealer and is dealt 2 cards face up, while the dealer is dealt 1 card face up and 1 card face down. Players in turn can draw another face up card if they wish, to get as close to 21 as possible, but can stop at any point. The dealer plays last and must draw to 17 and stand on 17 or more. If a player wins they receive double their stake and if they lose they lose all their stake. If a player has 2 cards with the same face value they can split them and receive another card for each original card and then play 2 separate hands. A player can also receive just 1 extra card and double their initial stake, this is called doubling down. It is usually the variations in when you are allowed to split and double down that vary the game of blackjack. Also if the dealer's face up card is an ace then insurance can be offered to the player. Insurance is a small side bet of half the initial bet that pays 2-1 betting that the dealer has blackjack.

For a full set of blackjack rules see chapter 2 *Beat The Dealer* by Edward Thorp. [1]

6.2.1 The Basic Strategy in Blackjack

When playing blackjack without any direct strategy i.e. playing on instinct the probability of winning is in favour of the house as with all casino games. This is usually by about 1-2% depending on which rules you are playing. This is due to the fact that the player loses as soon they are bust i.e. have more than 21 and the dealer wins as soon as they have blackjack (21 in 2 cards) regardless of the players cards.

The basic strategy aims to reduce these odds of winning to about evens. The basic strategy is a strategy which tells you whether to draw or stand, when to split and when to double down by looking at the dealers face up card. The increase in the players probability of winning comes by taking advantage of the fact that the dealer has to draw to 17 and stand on 17 or more, whereas players can stand on anything they want. The basic strategy does not count cards and hence takes each round individually, independent of previous hands.

The basic strategy is shown in the following tables;

You Have	Dealer Shows									
	2	3	4	5	6	7	8	9	10	A
19										
18										
17										
16									*	
15										
14									**	
13										
12										

soft standing number

hard standing number

* On Hard 16 with 2 cards i.e. (10,6) then draw, with 3 or more cards (6,2,3,5) then stand.

** Stand when holding (7,7)

This table shows when to draw and when to stand. A standing number is the number you need to reach before you stand. A hard hard is when you do not hold any aces and a soft hand is when you hold an ace e.g. (A,4) is soft 15 and (9,6) is hard 15. This shows that the strategy declares the player to stand on 12 when the dealer is showing a four. Although standing on hard 12 seems a rather incompetent move as any standing hand from the dealer causes him to win, the likelihood is that the dealer will bust and a draw from you could cause you to bust. The major differences between drawing/standing on hard or soft hands comes to the fact that drawing on hard 11 or less cannot make you hand worse and drawing on soft 16 or less cannot make you hand worse. Any hand less than 17 is equivalent to the player as if the dealer busts you win and if the dealer stands then he must have 17 or more so will win. This means that drawing on a soft 16 might turn your hand into a hard 15 or less, which would be equivalent to you, but it could also draw a better hand.

Now I will look at when it is the best strategy to split your hands. It is obviously not always best to split as you then play 2 hands and have the possibility to lose twice the stake.

You Have	Dealer Shows									
	2	3	4	5	6	7	8	9	10	A
A,A										
10,10										
9,9										
8,8										
7,7										
6,6										
5,5										
4,4										
3,3										
2,2										

Split

Do Not Split

It is usual when splitting aces that only 1 card is added per hand and you are not allowed to draw any more. Also if you hit a 10 it is counted as 21, but not blackjack.

The final tables for the basic strategy show when to double down. The table is split for hard doubling down in the first table and soft doubling down in the second table:

You Have	Dealer Shows									
	2	3	4	5	6	7	8	9	10	A
11	■	■	■	■	■	■	■	■	■	■
10	■	■	■	■	■	■	■	■		
9	■	■	■	■	■					
8				*	*					

■ Double Down

* Double Down except (6,2)

□ Do Not Double Down

You Have	Dealer Shows				
	2	3	4	5	6
A,7		■	■	■	■
A,6	■	■	■	■	■
A,5			■	■	■
A,4			■	■	■
A,3			■	■	■
A,2			■	■	■
A,A*				■	■

■ Double Down

* Double only if can't split

□ Do Not Double Down

I have not included a table of when to take insurance, this is because insurance is very rarely a bet that should be taken. For insurance to pay out you are making a bet that a 10 will appear. In a full deck of cards there are 16 tens and 36 non-tens. In an example if you take insurance 1300 times at £1 each then you should win 400 x 2 = £800, but lose 900 x 1 = £900, so you will be £100 down.

These tables explain the basic strategy, however they do not mention the order to which you play them. For example if you hold (4,4) against a 5, do you double down or split? Figure 6.2 shows the thought process for each hand.

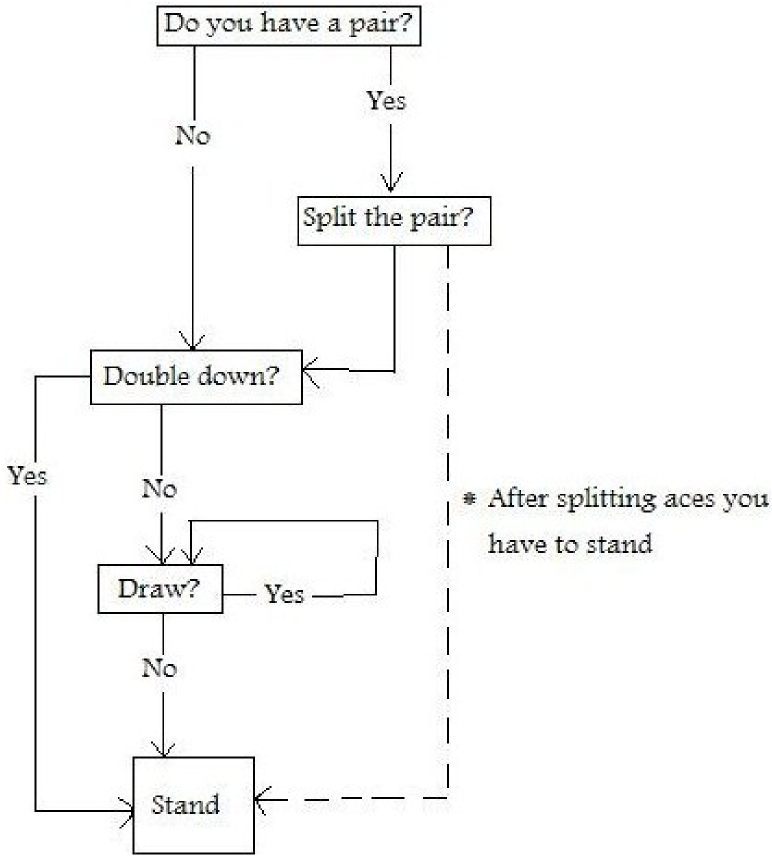


Figure 6.2 : The order of play for each hand of Blackjack

This is the basic strategy that brings the odds in favour of the house back to evens. It should always be followed, regardless of the size of your bet. Throughout the rest of this report, I have assumed that the basic strategy is being played.

6.2.2 Card Counting in Blackjack

The limitation of just using the basic strategy is that is assumes each round of betting is independent. However the key aspect of blackjack is that the cards are not shuffled between each round of betting. This makes it very different from most other casino games as it gives the player the chance to put the probability of winning firmly in favour of the player.

When certain cards in the deck are depleted this changes the probability of winning. For example imagine the first four cards are 4 aces then you know that until the cards are next shuffled an ace cannot appear. This means that no soft hands, blackjack or insurance can occur and this gives the house an advantage of 2.4% even when the player is using the basic strategy. Edward Thorp realised that removing different cards from the deck would vastly change the probability of winning and so throughout different rounds of betting the chance of winning would fluctuate between the player and the house. This can change by up to 100%. Imagine many rounds of betting have been played and the only cards left are two 7's and three 8's. Whatever you are dealt you will win by standing. If the dealer draws a (7,7) he must draw again, get an 8 and bust. If he gets (7,8) he must draw and either a 7 or 8 will bust him again. This shows the power of knowing what cards have gone and what cards are left. This is the essence of card counting.

After realising that certain cards when missing from the deck would produce more or less of an advantage, Thorp used the IBM 704 computer to help quantify the players edge over the casino. The following table shows when all the cards of each value are missing and the advantage to he player;

Table 6.1: [1]

Card Number	Advantage (%)
2	1.75
3	2.14
4	2.64
5	3.58
6	2.40
7	2.05
8	0.43
9	-0.41
10	-1.62
Ace	-2.42

This accentuates the point that low cards are good for the dealer and high cards are good for the player. This is because a dealer has to draw on stiff hands (12 - 16) so low valued cards allow him not to bust, whereas the player can stand on stiff hands. From this table it shows that when all fives have gone then the advantage to the player is 3.58%. This is actually the largest swing in an advantage for any of the cards. This is one of the simplest card counting strategies, the five count strategy.

Five Count Strategy

The five count strategy is simply counting the number of fives through different rounds of betting. When all fives gone through the deck, you will roughly have a 3.6% advantage and a favourable situation which you can exploit. However because this is very specific, only counting one number, then if you miss a 5 you can potentially miss out on a favourable situation and have to wait until the deck is shuffled and then you can start again.

An additional improvement to the five count is to count the total number of cards left in the deck. This does not have to be done exactly by counting every card seen, as after some practice a skilled played will accurately guess from the height of the deck how many cards are left to play. If the number of cards left is known and the number of 5's left is known as well then the ratio between them can tell you if the deck is five-rich or five-poor. At the start of the deck the ratio is 13 so if the ratio is less than 13 bet small and if it is bigger than 13 bet larger. If the number of fives becomes 0 then the ratio doesn't make sense but at this point you will have a 3.6% advantage so will be betting high anyway.

The five count system can be very successful. For example imagine you are playing alone against the dealer at 100 rounds per hour. On average about 10 hands per 100 all the fives will have gone and there will be a favourable situation. Also in this example playing the basic strategy gives the house an advantage of 0.2%. If we are betting £1 and £500 then on favourable situations we win £500x10x0.036=£180 and lose £1x90x0.002=18p on unfavourable situations. Hence per hour we will be making £179.82. However such large betting differences between your personal minimum and maximum bet can often cause too much attention and could be considered unwise.[1: page 56-57]

The Simple Point Count System

Although the five count strategy can be successful, you need to play a lot of hands for the favourable situations to arise. You also miss out on a lot of chances when you have the advantage by only counting 1 card. The next strategy is the simplest and easiest to use strategy that keeps track of the whole deck and how much of an advantage or disadvantage you have.

The simple point count system assigns either -1, 0 or +1 to each card throughout the deck. For cards (2-6) you assign +1, (7-9) you assign 0 and (10-A) you assign -1. From table 6.1 you can see that low cards give the dealer an advantage and high cards gives the player an advantage. Therefore when the count is high lots of low cards have gone and so are less likely to appear in the next round of betting. The opposite is true for when the count is low as lots of high cards have gone through the deck and will be less likely to come up. When the count is high you should bet more and when the count is low or negative you should bet the minimum. An easy betting strategy is to just bet the number of the count i.e the count is at +13 so bet 13 times the minimum bet.

This has a clear advantage over the five count strategy as you are continually changing your bet to the continuously changing probabilities. A similar improvement can be made to this strategy as I did with the five count. If the total number of cards left is known then it help you comprehend how strong your advantage is, for example if the count is +10 but there are still about 30 cards left, this is clearly not as strong a position as if you had only 10 cards left in which case you know all of them are (10-A). This simple point count strategy can also help you to decide if it could be profitable to take insurance. As I mentioned before it is rarely the optimal strategy to take insurance, however if the count is very high and there are a lot of 10's left in the pack then it can be the right decision.

These two physical strategies of card counting and trajectory tracking actively change the probability of winning to the favour of the player. Once this is achieved and a positive expectation has been found then the Kelly Criterion can be used to exponentially grow your wealth. In an example if you are using the five count strategy in blackjack and all the fives have gone then you have a 3.6% advantage. Therefore $a^* = 2p - 1 = 2(0.536) - 1 = 0.072$ and so you should bet 7% of you total capital if implementing the Kelly Criterion. The Kelly Criterion can also be used in roulette as long as a positive expectation is achieved. If you are betting on a single number and the probability of it occurring is 0.1, then $a^* = (35(0.1) - 0.9)/35 = 0.074$ and so you should bet 7.4% of your total capital.

These experiments with casino games became so successful and popular that casinos soon began changing the rules to try to discourage card counters and to make it harder to be successful. Card counting became far more popular than tracking the ball in roulette, this is obviously because card counting can be learnt reasonably easily whereas roulette requires a homemade, wearable computer. The number of times the deck was shuffled was increased and casinos stopped dealing to the last card. Also more decks can be used as this increases the house advantage. Different rule changes, change the advantage to the player in the following way:

Table 6.2: [17]

Card Number	Advantage (%)
Single deck	0
2 decks	-0.35%
4 decks	-0.52%
6 decks	-0.58%
8 decks	-0.61%
Dealer hits soft 17	-0.2%
Double on 9,10,11 only	-0.1%
Double on 10,11 only	-0.25%
No splitting aces	-0.18%

The effects of these disadvantages to the player are additive, so if you are playing a specific game when the house is using 6 decks, you can double down only on 10,11 and you can't split aces then the disadvantage to the player will start at 1.01%. It is obviously best to avoid casinos where they play these specific rules.

6.3 Further Applications in the Wider World

Thorp spent many years visiting casinos all across the world from Puerto Rico to Las Vegas, inventing slightly different counting strategies for blackjack for the different rules in each specific casino. However once he had realised the true power of the Kelly Criterion, he understood that he didn't need to keep travelling all over the world, risking his life as well as his money as many casinos had a strict policy on anyone caught counting cards. Thorp began using the Kelly Criterion on the stock markets and over the next 30 years placed over 80 billion dollars worth of trades.

Chapter 7

Conclusion

In this report I have discussed many of the problems and solutions of how to try to beat the casinos and make a positive growth in your capital. For the majority of casino games, a positive expectation cannot be achieved otherwise the casino would not offer the game. However in some games, especially blackjack physical strategies such as card counting can give you a positive expectation. Once a certain strategy has given you a positive expectation then the Kelly Criterion will tell you the optimal fraction of your capital to bet in order maximise you expected growth and minimise your chance of gamblers ruin.

I introduced the idea of very simple stopping strategies by way of an example to help describe the aims for later strategies. If a strategy has a clear notion of where to stop, then the player can set a fixed amount of profit that they wish to receive. From chapter 2 you can see that the more you wish to win the lower the chance of reaching that amount. This explains the concept of risk verses reward which is key to any betting strategy.

The specific betting strategy that I focused on in this report was the Kelly Criterion. I looked at the derivation of the actual Kelly Bet and the rate of growth. I then moved on to seeing how variations of the Kelly Bet affect the rate of growth and whether any of these variations could be beneficial in any way. I realised that when betting a higher proportion than the Kelly Bet, it was possible to initially receive a faster rate of growth, but the chance of gamblers ruin was significantly larger. This was apparant in the simulation I made as when betting double the Kelly Bet gamblers ruin was achieved after only 40 rounds of betting. I continued to outline other betting strategies and compare them to the Kelly Criterion. Many of the other progressive betting strategies seemed to be less applicable in the real world because of the rigid nature of the strategies. There are many rules in casinos that would render many of these strategies useless including maximum or minimum betting limits. When comparing these strategies with the Kelly Criterion I decided that the Kelly Criterion can be considered a much more successful strategy as it gives a much higher expected rate of growth. This was also seen in another simulation I created comparing the expected capital of all the different strategies after 500 rounds of betting. It is also a much more adaptable strategy as it compensates for a varying game. As the formula includes the probability of winning, when the probability is changing, the fraction of your capital that the Kelly Bet instructs you to bet changes as well.

The simulations that I set up in chapters 3 and 5 I used Microsoft Excel to compute them. I set up a random number generator to create when the player the won or lost and then using different formulae to compute the capital after each round of betting for each betting strategy. The tables in chapters 4,5 and 6 were also constructed using Microsoft Excel unless otherwise stated.

In the final chapter I discussed how the strategies I have explained can be used in real world casinos. Overall in the game of blackjack the most successful strategy would be to employ a mixture of different strategies that work together. Generally you should use the basic strategy to bring the probability of winning back to evens so you will not lose too much before a positive expectation opportunity is found. At the the same time as this a simple point count should be undertaken to maximise the number of opportunities of positive expectation. Once one of these opportunities arise then you should place a bet according to the Kelly Criterion. It was this way that Edward Thorp became so successful in breaking the casinos.

A very important point to remember is that you should never play with money that you cannot afford to lose. Apart from the obvious risk of losing all of your money, the added psychological pressure puts strain on the betting strategy you employ. It is likely to cause you to be more erratic and make the betting strategy fail. On the other hand if you are playing with money that you can afford to lose then this will lead to confidence and accurate play that is more likely to end in success.

There are many more detailed card counting schemes that would further increase the benefit to the player. However as with any of the schemes described in this report, none of them give any certainty of success and there is still some risk to be taken. It is for this reason that gambling should be undertaken very cautiously and the use of some of the mentioned strategies will neither prevent gamblers ruin nor give a definitive success, but they can act as a preventative measure to ensure that some of the players goals are satisfied.

Word Count 13920

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